Computing Submesh Reliability in Two-Dimensional Meshes

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Abstract

An analytical model for computing submesh reliability is proposed in this paper. The system is considered operational as long as a functional submesh of the required size is available. We use the principle of inclusion and exclusion to find the exact probability of having a functional submesh within a partition of the mesh. The reliability of a partition is then used to approximate the submesh reliability of the system and thus this model is called partitioned mesh (PM) model. The partitions are taken along either dimension of the mesh. The selection of the dimension on which the partitions are made is based on the required submesh size. It is shown that the submesh reliability in every partition reduces to a polynomial with only two terms. The partitioned mesh model is validated with simulation and compared with the earlier proposed approximation techniques. It is shown that the PM model provides better approximations for submesh reliability with constant computational complexity.

INDEX TERMS: Column Partition, Partitioned Mesh Model, Row Partition, Submesh Reliability, Two Dimensional Mesh.

I. Introduction

The mesh-connected processor array is a popular architecture used in parallel processing. Existing examples of such systems include the Illiac IV, MasPar, Touchstone Delta, Intel Paragon [1].

The fault-tolerance of mesh connected systems has been studied in [2-4]. Earlier works have primarily focused on the design and analysis of reconfigurable systems. In [2, 3], a fault causes the elimination of either an entire row or column from the mesh, and the minimum working configuration consists of a single functional row or column. These techniques are efficient for reconfiguring systolic systems. For general applications and with today's high performance systems designed with powerful and expensive processing nodes, the removal of functional nodes becomes undesirable.

Mesh connected parallel computers are used in a partitionable environment where a task requires only a certain number of processing elements arranged in adjacent locations and does not require an entire mesh to perform its function. The task requirements are specified in terms of submeshes. Thus in a submesh reliability model, the system is considered operational as long as a submesh of the required size is available. Analyzing the reliability for submesh helps the system designer to understand the reliability of the mesh system in performing a particular task.

Submesh reliability is dependent on the processor allocation schemes used. Several submesh allocation schemes have been proposed in the literature [5-7]. These allocation schemes vary in terms of the submesh recognition ability and thus result in different submesh reliability. The two-dimensional buddy system (TDBS) [5] and the frame sliding (FS) method [6] allocate submeshes at specific locations in the system. As indicated by Zhu [7], these two methods have the problems of fragmentation, and may cause allocation misses. Zhu proposed an algorithm, which is capable of allocating submeshes at any possible location provided it exists.

As the possible submesh locations in TDBS and FS do not overlap, the analysis is straightforward. Exact models were derived in [4] for TDBS and FS. In case of the submesh allocation with perfect recognition (Zhu's schemes [7]), the possible submesh locations overlap and thus the analysis becomes extremely difficult. Earlier attempts to analyze the exact reliability demonstrated the difficulty and have left it as an open problem [8,9]. Two approximation models were derived in [4] to give the lower bounds for mesh systems with perfect recognition ability -- the expanding row/column technique, and the row-folding technique. The expanding row/column technique gives a tight bound when the degradation factor is low (less than 50%), but gives a pessimistic estimation of the submesh reliability when the degradation factor is higher. On the other hand, the row folding technique gives a better approximation at high degradation but is not as good in predicting the submesh reliability at low degradation. The main drawback of both these approximation techniques is the computational complexity. Both techniques require a recursive algorithm to calculate the consecutive k-of-n reliability. The computational complexity of calculating the consecutive k-of-n system is non-polynomial and thus is only suitable for small system size.

In this paper, we analyze the submesh reliability of mesh system with perfect recognition ability. The motivation behind this work is to find an approximation technique that can provide a better estimation of the submesh reliability with a reasonable computational complexity so that it can be used for any system size. An exact model for the mesh system with perfect recognition ability can be derived with the principle of inclusion and exclusion. However, the time complexity involved is too high and thus makes this approach unacceptable. By decomposing the system into smaller partitions along one dimension, the submesh reliability using the principle of inclusion and exclusion within each partition is reduced to a deterministic polynomial with only two terms. The partitions can be made along either dimension of the mesh. The submesh reliability of the entire system can then be approximated by treating the system as a parallel system consists of these partitions.

In the next section, we define the parameters and assumptions used in deriving our model. Section III explains the detail of the polynomial model for the partitioned mesh technique. The model is then validated and compared with previous models in Section IV, followed by the concluding remarks in Section V.

II. Preliminaries

We consider an MxN mesh connected system where M is the number of rows and N is the number of columns of processing elements. An incoming job is assumed to require a submesh of size mxn, where m≤M and n≤ N. Submesh reliability of the mesh is defined as the probability of being able to allocate a fault-free submesh of the required size for the execution of a particular task. The proposed analytical model is based on the following assumptions:

(i) A node failure does not affect the normal operation of any other node.

(ii) The distributions of failure time of the nodes are assumed exponential in nature and are independent of each other.

(iii) The model assumes that the allocation algorithm is always able to recognize an operational submesh of the required size if it exists.

Failure of a node includes the failure of the processing element and the communication links associated with it. The failure rate for an individual node is denoted as λ. The reliability of a single node is defined as a function, $R(t) = e^{-\lambda t}$, with respect to time t and its failure rate λ. Coverage factor, denoted as C, is the probability that a system successfully detects a fault and is recovered after a fault has occurred. In the case of an uncovered failure, the status of the system is unpredictable and considered faulty.

III. Submesh Reliability Model by the Principle of Inclusion and Exclusion

In this section, we describe the application of principle of inclusion and exclusion in finding the exact submesh reliability model.

A. *Principle of Inclusion and Exclusion in Submesh Reliability*

In an MxN mesh, there are $X = (M - m + 1) \times (N - n + 1)$ possible locations to locate a submesh of size mxn. For the clarity of explanation, we name these submeshes by their upper left-most nodes. So we have submeshes located at positions (1,1) through (M-m+1, N-n+1). The probability of having a mxn submesh at any specific location is exactly $R^{mn}(t)$. The submesh reliability is equal to the probability of having at least one of the X submeshes consisting of functional nodes. Having a functional submesh at any single location is a probabilistic event and the submesh reliability equals to the probability of occurrence of these events. This probability can be calculated using the principle of inclusion and exclusion as

$$\begin{aligned} R_{sys} = &\sum \mathit{Prob\{one\ submesh\ exists\}} - \\ &\sum \mathit{Prob\{any\ two\ submeshes\ coexist\}} \\ &+ \sum \mathit{Prob\{any\ three\ submeshes\ coexist\}} - \cdots \\ &+ (-1)^{X-1} \sum \mathit{Prob\{X\ submeshes\ coexist\}}. \end{aligned}$$

The probability of having more than one submesh functional can be found as $R^{Y}(t)$, where Y is the number of nodes required for all the coexisting submeshes. Depending on the locations of the coexisting submeshes concerned, Y can be different for the same number of coexisting submeshes. Fig. 1 shows such an example for two coexisting submeshes. The two 3x3 functional submeshes at location (1,1) and (2,1) requires 12 nodes to be functional, namely, nodes (1,1) to (4,3). On the other hand, for two 3x3 submeshes to coexist at location (1,1) and (2,2), 14 nodes have to be operational. The above example shows that in the second summation of our principle of inclusion and exclusion formula, among others, the terms $R^{12}(t)$ and $R^{14}(t)$ exist. Once all Y's are found for every possible submesh combination and different number of coexisting submeshes, the complete submesh reliability model can be constructed using the principle of inclusion and exclusion.

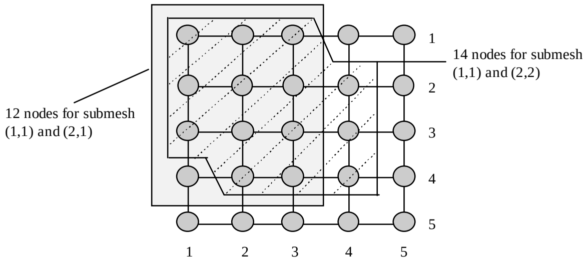

Fig. 1. Number of Nodes Required for Submeshes at Different Locations.

To obtain the complete reliability polynomial, two things need to be analyzed. First, a submesh list for different number of coexisting submeshes has to be generated. Second, the number of required nodes Y for every submesh combination from the submesh list has to be determined. For X possible locations of functional submeshes, there are 2^X combinations. Since we need to check over each of these combinations to determine the number of required nodes for all the submeshes in the combination to be operational, this approach have a complexity of $O(2^X)$. This kind of complexity becomes unacceptable even with a small increase of X.

B. Partitioned Mesh (PM) Model

To reduce the complexity of computations, we use an approximation technique based on dividing the entire mesh into non-overlapping partitions. We call this technique as Partitioned Mesh (PM) model.

There are two different ways to partition the system. Consider a task requirement of mxn submesh in an MxN mesh. The first approach considers a partition of m rows and 2n columns. If 2n is more than N, we make the partition' s size mxN. The partitions thus formed are called row partitions (RPs). The second approach considers partitions of 2m rows and n columns. If 2m is more than M, then we make the partition size Mxn. The partitions thus formed are called column partitions (CPs). Therefore an MxN mesh can be partitioned into $\left\lfloor \frac{M}{m} \right\rfloor \times \left\lfloor \frac{N}{2n} \right\rfloor$ RPs or $\left\lfloor \frac{M}{2m} \right\rfloor \times \left\lfloor \frac{N}{n} \right\rfloor$ CPs, where 2m≤M and 2n≤N. When 2m≥M the number of RPs is equal to $\left\lfloor \frac{N}{n} \right\rfloor$ and when 2n≥N the number of CPs is equal to $\left\lfloor \frac{M}{m} \right\rfloor$. Note that while making the RPs or CPs, there may be cases where not all the nodes are covered. The number of uncovered nodes adds to the error in the model.

There are two factors that should be considered while making the decision whether to make CPs or RPs. First, the number of uncovered nodes should be minimized. The uncovered nodes are ignored in our analytical model and thus contribute to the error. Second, the number of partitions should be minimized. As we are considering non-overlapping partitions, all the submeshes that overlap two or more partitions are ignored in our analysis and contribute to the error.

By extending an mxn submesh into an mx2n RP, there will be X = n+1 possible locations to locate a functional submesh. An example of a 3x8 RP for the 3x4 submeshes is shown in Fig. 2. By extending the 3x4 submesh into 3x8 RPs, the possible locations for a functional submesh of the required size in a partition are equal to 5. One important property for the possible submesh locations in a partition is each of these submesh locations overlaps with or is adjacent to another submesh location. This property plays an important role in simplifying the submesh reliability within each RP or CP.

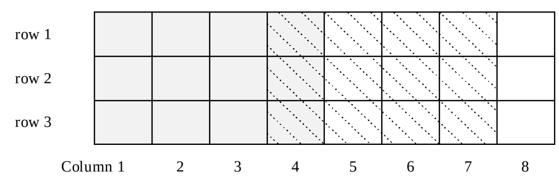

Fig. 2. A (3x8) Row Partition for Finding the Reliability of (3x4) Submeshes.

The submesh reliability is a polynomial of the single node reliability, R(t). The polynomial can be constructed using the principle of inclusion and exclusion to find the coefficients. The exponents of the polynomial are found by counting the number of required functional nodes for different combination of possible submesh locations. In the example shown in Fig. 2, there are X=5 possible locations for submesh of the required size. The reliability polynomial consists of $R^{Y_i}(t)$, where Y_i is between 12 and 24, and is a multiple of 3. Twelve and twenty-four are the smallest and largest possible number of required functional nodes for different submesh combinations. The height of this RP is equal to the height of the submesh required, so at least an entire column has to be added into consideration when more submeshes are considered. The width of the partition is taken as twice the submesh width, so any two submeshes in this partition will overlap with or be adjacent to one another. Therefore, no matter what submesh combination we are considering, the required functional nodes for these submeshes will be in consecutive columns. The number of processing nodes in these consecutive columns, Y, represents the exponent that appears in the reliability polynomial. The number of submeshes determines the sign of the coefficient of $R^Y(t)$ in the reliability polynomial.

Table 1 lists the coefficients of every term generated by including different number of events for the existence of an mxn submesh in an mx2n RP. The derivation of the reliability for a CP is similar and thus we use only the RP to show the derivation. The sum of each column in Table 1 represents the coefficient for a different term in the reliability polynomial. The coefficient of degree *km* is affected by possible locations of *k* consecutive operational columns and these *k* consecutive columns can be assigned to different number of functional submeshes. Rows in Table 1 show how different number of submeshes can contribute to terms with different exponents. In other words, an entry in the column R^{km} and row *j* shows how *j* functional submeshes can be selected with *k* consecutive operational columns. Table 1 lists only coefficients from R^{mn} to $R^{(n+4)m}$. The actual polynomial consists of terms with exponents upto R^{2mn}.

Because n consecutive columns in a mesh can be assigned to only one functional submesh and n+1 consecutive columns can only be assigned to two consecutive submeshes, the first two coefficients show how n and n+1 consecutive operational columns can be selected. For all the other entries in Table 1, there are two combinatorial terms. The first term is the number of ways one can choose the consecutive columns (rows) of working nodes in the mesh. After the consecutive columns (rows) of working nodes have been chosen, it is possible to determine the submesh combinations that require this number of consecutive columns (rows). It can be observed that *n+k* consecutive columns (*k* between 2 and n) can be caused by combinations of *2* to *k+1* submeshes. We can visualize these *n+k* consecutive columns as having at least two boundary submeshes which will require all these columns to be operational. Beside these two boundary submeshes, the consecutive columns can accommodate upto *k-1* other submeshes without changing the exponent (number of required nodes) in the reliability polynomial. The second combinatorial term determines how many different ways can these other *k-1* submeshes be selected.

Associating the sign of the coefficients with the second combinatorial term in Table 1, the sum of all the second terms in the same column for coefficients of R^{mn+km} is equal to the binomial $[1+(-1)]^{k-1}$, except when k is equal to 0 and 1. This binomial function is obviously equal to zero and thus the coefficients of all the terms in the reliability polynomial are zero except the first two terms. The coefficient for R^{mn} is equal to X, and the coefficient for R^{mn+m} is equal to X-1. A reliability polynomial with only two terms can thus be simply derived for an RP as,

$$R_{RP}(t) = X \cdot R^{mn}(t) - (X-1) \cdot R^{mn+m}(t)\,. \tag{1}$$

Coverage factor of a failure can be incorporated into the PM reliability model. Each term, $R^Y(t)$, means that Y functional nodes are required for the coexisting submeshes. Among the remaining nodes, any number of failures will not affect the operation of the system as long as these failures are covered failures. Let α be the number of nodes in a partition. It is equal to 2mn when the size of partition considered is mx2n as in RPs or 2nxm as in CPs. If the partition size is mxN or Mxn, α is equal to Mn or mN, correspondingly. The probability of not having any uncovered fault among the other

Table 1. The Coefficients of the Reliability Polynomial.

# of submeshes	R^{mn}	$R^{(n+1)m}$	$R^{(n+2)m}$	$R^{(n+3)m}$	$R^{(n+4)m}$ ....
1 (+)	$\binom{X}{1} = X$				
2 (-)		$\binom{X-1}{1} = X-1$	$\binom{X-2}{1}\times\binom{1}{0}$	$\binom{X-3}{1}\times\binom{2}{0}$	$\binom{X-4}{1}\times\binom{3}{0}$
3 (+)			$\binom{X-2}{1}\times\binom{1}{1}$	$\binom{X-3}{1}\times\binom{2}{1}$	$\binom{X-4}{1}\times\binom{3}{1}$
4 (-)				$\binom{X-3}{1}\times\binom{2}{2}$	$\binom{X-4}{1}\times\binom{3}{2}$
5 (+)					$\binom{X-4}{1}\times\binom{3}{3}$
....					
X $(-1)^{X-1}$					

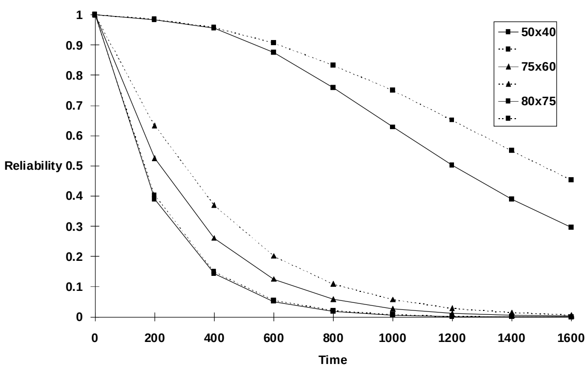

Fig. 3. Submesh reliability for a (100x80) mesh. λ=0.000001, C=0.99.

nodes can be calculated by the summation, $f(Y)=\sum_{i=0}^{\alpha-Y}\binom{\alpha-Y}{i}R^i\{C(1-R)\}^{\alpha-Y-i}$. This summation has to be multiplied with $R^Y(t)$ in the reliability polynomial derived when coverage factor is considered. Thus the reliability polynomial for an RP with coverage factor incorporated in Equation (1) is equal to

$$R_{RP}(t)=X\cdot R^{mn}(t)\cdot f(\alpha-mn) \\ -(X-1)\cdot R^{mn+m}(t)\cdot f(\alpha-mn-m). \tag{2}$$

The submesh reliability of the entire mesh can be approximated as a parallel system consisting of the RPs. Let S be the number of RPs in the system. The system reliability can be modeled as,

$$R_{sys}=1-(1-R_{RP})^S. \tag{3}$$

The reliability polynomial can be obtained for a CP by substituting the corresponding variables in the RP model.

IV. Results and Discussion

A. Model Validation

The partitioned mesh model for submesh reliability is compared with the simulation results. A mesh of size (100x80) is considered. Task requirements of three different submesh sizes are analyzed, namely 50x40, 60x75, and 75x80. The simulation is run for 10,000 iterations. Fig. 3 illustrates the system reliability of the three different submesh requirements. The dotted lines represent the simulation results, and the solid lines are the analytical results for the corresponding submesh sizes. The results show the accuracy of the analytical model. This model gives good approximation at low degradation, such as in the case of 80x75 and 60x75 submeshes. For higher degradation, 50x40, the model still gives a reasonable approximation with small time t. The approximation becomes worse as the time grows and the system reliability falls. However, it is unlikely that the system will be operated with reliability below certain values, or with such high degradation.

B. Comparison and Discussion

Table 2 lists the submesh reliability predicted by the PM model and the expanding R/C technique [4]. The system and submesh sizes compared are the same ones used in the previous subsection. These submesh requirements represent the high degradation to low degradation scenarios from 75% to 25%. It can be inferred from Table 2 that the PM model gives tighter bounds for the 50x40 case. It gives a bound of 6% higher at time 200, and about 56% higher at time 1000 than the expanding R/C model. For the low degradation submesh (75x60), the PM model gives more conservative approximations. The difference is very insignificant (around 0.4%) at time 200. The largest difference obtained is at about 3.6 % at time 1000. The submesh reliability at that time is only around 0.03. It is very unlikely that a system will be operated with such a low reliability. For the submesh with the lowest degradation (80x75), both models give identical results.

Table 2: Comparison of R/C extension technique with the partitioned mesh (PM) model.

	50x40		**75x60**		**80x75**	
Time	R/C	PM	R/C	PM	R/C	PM
0	1.000	1.000	1.000	1.000	1.000	1.000
200	0.938	0.995	0.527	0.525	0.389	0.389
400	0.807	0.958	0.262	0.261	0.143	0.143
600	0.660	0.875	0.126	0.125	0.051	0.051
800	0.521	0.758	0.059	0.058	0.018	0.018
1000	0.402	0.628	0.028	0.027	0.006	0.006
1200	0.304	0.502	0.012	0.012	0.002	0.002
1400	0.226	0.389	0.005	0.005	0.001	0.001

For system with high degradation, row folding technique is chosen over expanding R/C technique for its better accuracy. Figures 5 and 6 compare the submesh reliability estimated by row folding technique and the PM model with high degradation. Different failure rates are used in these two figures to make the difference between the two models prominent. In the two cases compared, we have shown that the PM model provides an even tighter bound than the row folding technique. Unlike the expanding R/C and row folding techniques, which only provide good approximation of the reliability with certain range of degradation factor, the PM model analyzes submesh reliability with consistent accuracy for any degradation factor.

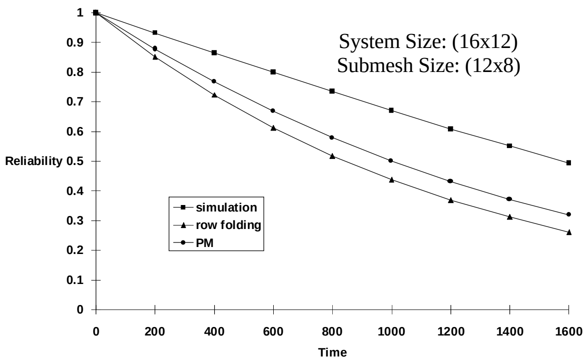

Fig. 4. Submesh Reliability with 50% degradation. C=1.0, λ=0.00001.

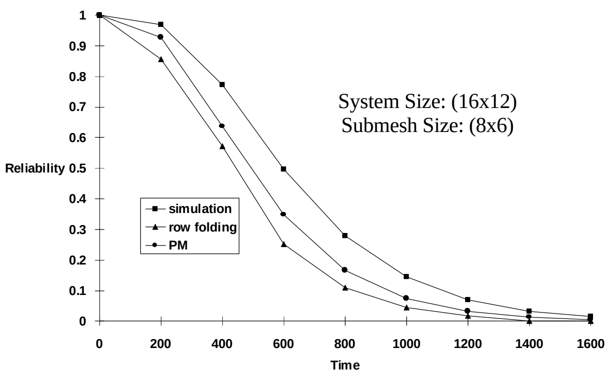

Fig. 5. Submesh Reliability with 75% degradation. C=1.0, λ=0.0001.

The most important feature about this new model is its simplicity. Both the previously proposed techniques require a recursive algorithm to carry out the calculation for the consecutive k-of-n system. The computational complexity of the recursive algorithm is non-polynomial and thus requires tremendous computing time even for a small system size. In our new model, the submesh reliability in every partition is a polynomial with only two terms. Thus, the complexity for obtaining the reliability polynomial for every RP or CP is constant. The approximation of parallel system with RPs and CPs is also of constant complexity.

V. Concluding Remarks

This paper discusses the submesh reliability evaluation techniques in mesh connected multiprocessors. A partitioned mesh (PM) model is introduced for finding the mesh reliability using the perfect submesh recognition allocation algorithm. The entire mesh is first divided into several partitions based on the required submesh size. Each of these partitions is analyzed using the principle of inclusion and exclusion. Reliability of the partitions are then used to derive the system reliability using the parallel system model. The model is validated through simulations. Two major improvements over the previous techniques are obtained in the PM model. First, previous models provide good bounds only within certain degradation. The PM model provides consistent accuracy with a wide range of degradation. Second, the exact analysis of submesh reliability is known to be extremely difficult [8,9]. Previous approach on approximating the submesh reliability is highly complex and does not provide a tight bound. The proposed PM model is very simple and also provides a tighter bound than that of [4]. The model improves the complexity from a non-polynomial complexity as used in [4] to a constant. These two advantages make the partitioned mesh model favorable.

References

[1] K. Hwang, "Advanced Computer Architecture: Parallelism, Scalability, Programmability." McGraw Hill, 1993.

[2] J. A. B. Fortes and C. S. Raghavendra, "Gracefully Degradable Processor Arrays," IEEE Trans. on Computers, vol. c-34, No. 11, pp. 1033-1044, Nov. 1985.

[3] N. Lopez-Benitez and J. A. B. Fortes, "Detailed Modeling and Reliability Analysis of Fault-Tolerant Processor Arrays," IEEE Trans. on Computers, vol. 41, No. 9, pp. 1193-1200 September 1992.

[4] P. Mohapatra and C. R. Das, "On Dependability Evaluation of Mesh Connected Multiprocessors," to appear in the IEEE Trans. on Computers, Sept. or Oct. 1995.

[5] K. Li and K. H. Cheng, "Job Scheduling in a Partitionable Mesh Using a Two Dimensional Buddy System Partitioning Scheme," IEEE Trans. on Parallel and Distributed Systems, pp. 413-422, October, 1991.

[6] P. J. Chuang and N. F. Tzeng, "An Efficient Submesh Allocation Strategy for Mesh Computer Systems," Int. Conference on Distributed Computing Systems, pp. 256-263, May, 1991.

[7] Yahui Zhu, "Efficient Processor Allocation Strategies for Mesh-Connected Parallel Computers," Journal of Parallel and Distributed Computing 16, pp. 328-337, 1992.

[8] A. A. Salvia and W. C. Lasher, "2-Dimensional Consecutive-k-out-of-n:F Models," IEEE Trans. on Reliability, vol. 39, No. 3, pp. 382-385, August, 1990.

[9] T. K. Boehme, A. Kossow, and W. Preuss, "A Generalization of Consecutive k-out-of-n:F systems," IEEE Trans. on Reliability, pp. 451-457, September, 1992.